A Markov jump process approximation of the stochastic Burgers equation*

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Abstract

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We consider the stochastic Burgers equation

$$\frac{\partial}{\partial t}\psi(t,r) = \Delta\psi(t,r) + \nabla\psi^2(t,r) + \sqrt{\gamma\psi(t,r)}\eta(t,r) \tag{1}$$

with periodic boundary conditions, where $t \geq 0$, $r \in [0,1]$, and η is some space-time white noise. A certain Markov jump process is constructed to approximate a solution of this equation.

1 Introduction

Scientific and engineering systems are often subject to uncertainty or random influence. Randomness can have delicate impact on the overall evolution of such systems. Taking stochastic effects into account is of central importance for the development of mathematical models of complex phenomena in engineering and science. Macroscopic models in the form of partial differential equations for these systems contain such randomness as stochastic forcing, uncertain parameters, random sources or

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inputs, and random boundary conditions. Stochastic partial differential equations (SPDEs) are appropriate models for randomly influenced systems.

Most of stochastic partial differential equations models are nonlinear in nature. Especially the Burgers equation with stochastic noise has attracted considerable attention, for example, as a simplified model of fluid turbulence. Due to the nonlinearity, numerical simulations are often necessary in order to understand the dynamical behavior of the stochastic models. In this paper, we propose a Markov chain approximation method for a stochastic Burgers equation and prove its convergence.

More explicitly, our reaction-diffusion model is constructed by dividing the unit interval into N cells of length $1/N$. We place an initial distribution of approximately Nl particles into the cells. The particles in each cell independently jump to neighboring cells according to Poisson processes with rates $N^2 n_k$ where n_k is the number of particles in cell k, and are born or die with rates $\gamma N l n_k/2$. Moreover, to obtain the desired nonlinearity we allow particles to jump to the cell next to them on the left-hand side at a rate approximately given by $N n_k^2/l$. Our approximating process is given by a step-function-valued process X^N defined by the rescaled "densities" n_k/l. We mainly assume that $l \geq cN$. Then we show that for $N \to \infty$ there exists a limit satisfying (1).

Our method bases on a work by D. Blount [10]. He obtained a process solving the SPDE

$$\frac{\partial}{\partial t}\psi(t,r) = \Delta\psi(t,r) - d\psi^2(t,r) + \alpha\psi(t,r) + \sqrt{\psi(t,r)}\eta(t,r). \tag{2}$$

(where $d \geq 0$ and η is some space-time white noise) as a high-density limit of a Markov jump process consisting of birth- and death-processes and diffusion processes similar to the jump process described above. We verify some important martingale relationships between the approximating Markov jump process and its generator by the method of [18]. This allows writing the process approximately as

$$X^N(t) - X^N(0) = \int_0^t \Delta_N X^N(s) + \nabla_N^+ (X^N(s))^2 ds + Z^N(t)$$

where Z^N is a mean-0-martingale and Δ_N and ∇_N^+ are a discretized Laplace operator and a discretized first derivative, respectively. It turns out that Z^N consists of a part originating from diffusion and a part coming from the birth process, where the diffusion part vanishes in the limit. By the method of [10] we can show tightness of the reaction part of Z^N in spaces $D(0,T,H^\alpha(0,1))$ with Skorohod metric, where $H^\alpha(0,1)$ are certain Sobolev spaces. To show tightness of the remaining part of the approximating process X^N in $L^2(0,T,L^2(0,1))$ we adapt a method of [13] and [34] and especially show a discretized version of the compactness result [19]. The representation of the limit, which is now inferred from the theorems of Prokhorov and Skorohod, as solutions of (1) follows by an application of the theory of super Brownian motion, see [15], [35], and [24].

Markov jump process approximations of reaction-diffusion equations have been studied for a long time. A deterministic reaction-diffusion equation with polynomial nonlinearities is treated in [2]. The approximation of a linear reaction-diffusion equation by space-time jump Markov processes is investigated by D. Blount and P. Kotelenez e.g. in [27], [28], and [6], for various assumptions on the initial density of

particles and the number of cells, and in different function spaces, and central limit theorems are proved. These results are generalized to reaction-diffusion equations with polynomial nonlinearities in [29], [7], and [8] by these authors. In [9] laws of large numbers in a high density and in a low density limit and a central limit theorem is given for Equation (2) without noise. Only recently, M. Kouritzin and H. Long [30] generalized the ansatz to a much broader class of nonlinearities and applied the idea to a reaction-diffusion equation that is driven by a Poisson point process and describes water pollution. However, their nonlinearities do not involve spatial derivatives. Our work seems to be the first step in this direction.

When our work was almost finished, we learned about a preprint by G. Bonnet and R. Adler, [11], where Equation (1) is studied on the entire real line. Their approach is based on a multidimensional stochastic differential equation driven by (multiplicative) white-in-time noise. By means of Green function representation and a tightness argument convergence of a subsequence of solutions of the approximating SDE towards a solution of (1) is shown.

Moreover, the classical Burgers equation has been investigated in the probability literature in a number of ways, e.g. as limit of an asymmetric simple exclusion process or as limit of certain particle systems driven by Brownian motions. We cannot give a complete survey on the vast literature in this field. See e.g. [3], [12], [16], [17], [23], [31], and [33], just to name a few. An approximation of the 2-D-Navier-Stokes equation is found in [32].

Our work is organized as follows. In Section 2, we construct the Markov chain approximations to (1) in the manner of the above mentioned works. Section 3 contains the proofs of these results and in Section 4 we establish some auxiliary results.

2 Problem and Result

In this section we introduce our models and present the main result.

The stochastic model: is the stochastic Burgers equation

$$\begin{aligned}&\frac{\partial}{\partial t}\psi(t,r) = \Delta\psi(t,r) + \nabla\psi^2(t,r) + \sqrt{\gamma\psi(t,r)}\eta(t,r),\\ &\psi : [0,T]\times[0,1] \to \mathbb{R},\end{aligned} \tag{3}$$

with initial condition $\psi(0,r) = \psi_0(r)$ and periodic boundary conditions. η is some space-time-white noise, and Δ and ∇ denote $\frac{\partial^2}{\partial r^2}$ and $\frac{\partial}{\partial r}$, respectively.

The approximation model: is a Markov jump process defined as follows. Divide $[0,1]$ into N cells of width $1/N$. $[0,1]$ is from now on is identified with a circle of circumference 1, to obtain periodic boundary conditions. We place an initial distribution of approximately Nl particles into the cells, corresponding to the initial conditions given in the sequel, so l can be seen as initial average number of particles in a cell. For $1 \le k \le N$ and $t \ge 0$ let $n_k^N(t)$ be the number of particles in cell k at time t. We suppress the l-dependence of n_k^N in our notation.

Let $n^N(t) = (n_1^N(t), \ldots, n_N^N(t))$ in $\mathbb{N}_0^N$. ($\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$.) Define the jump rates for $n^N(t)$ by

$$
\begin{aligned}
(n_{k-1}, n_k) &\rightarrow (n_{k-1}+1, n_k-1) \\
&\qquad \text{at rate } N^2 n_k + \frac{N}{3l}(n_k^2 + n_k n_{k-1} + n_{k-1}^2), \\
(n_k, n_{k+1}) &\rightarrow (n_k-1, n_{k+1}+1) \quad \text{at rate } N^2 n_k. \\
n_k &\rightarrow n_k+1 \quad \text{at rate } \gamma N l n_k/2. \\
n_k &\rightarrow n_k-1 \quad \text{at rate } \gamma N l n_k/2.
\end{aligned}
\tag{4}
$$

(Observe the periodic boundary conditions for n^N, i.e. $n_{k+zN} = n_k$, $z, k \in \mathbb{Z}$.) For an introduction in Markov jump processes see, e.g., [18]. The state space of the process is $E := \mathbb{N}_0^N$.

The generator of the process $n^N(t)$ is given by

$$
A(i,j) = \begin{cases} -\lambda(i) & i = j \\ \lambda(i)Q(i,j) & i \neq j, \end{cases} \tag{5}
$$

where i, j are elements of the state space E. Let $i = (n_1, \ldots, n_N)$, then $\lambda(i)$ is the sum over the rates in (4), $\lambda(i) = \sum_{k=1}^N N^2 n_k + \frac{N}{3l}(n_k^2 + n_k n_{k-1} + n_{k-1}^2) + N^2 n_k + \gamma N l n_k$. The time the process remains in state i until the next jump is exponentially distributed with parameter $\lambda(i)$. $Q(i,j)$ is the transition function of the underlying Markov chain corresponding to the states of the process. If state j can be reached from state i, then $Q(i,j) = \text{rate}(i,j)/\lambda(i)$, otherwise $Q(i,j) = 0$. If, for instance, $j = i$ up to a jump of one particle from a cell to a neighboring cell, that means $j - i = (0, .., 1, -1, 0..)$ for instance, where the -1 is at position k then $\text{rate}(i,j) = N^2 n_k + \frac{N}{3l}(n_k^2 + n_k n_{k-1} + n_{k-1}^2)$. By

$$
Af(i) = \sum_{j \in E} (f(j) - f(i))A(i,j) \tag{6}
$$

A operates on the real valued functions $f : E \to \mathbb{R}$, see [18]. From [18], Prop. 4.1.7, e.g., we obtain that $f(n^N(t)) - \int_0^t Af(n^N(s))ds$ is a martingale w.r.t. the filtration $\mathcal{F}_t^N \subset \mathcal{F}$ on the underlying probability space $(\Omega, \mathcal{F}, P)$ which is the completion of the σ-field induced by the process $n^N(t)$. Let $f = f_k$, $f_k(n_1, \ldots, n_N) = n_k$, then

$$
\begin{aligned}
\text{with } I(s) := {} & N^2(n_{k+1}(s) - n_k(s)) - N^2(n_k(s) - n_{k-1}(s)) \\
& + \frac{N}{3l}(n_{k+1}^2(s) + n_{k+1}(s)n_k(s) + n_k^2(s)) \\
& - \frac{N}{3l}(n_k^2(s) + n_k(s)n_{k-1}(s) + n_{k-1}^2(s)), \\
n_k^N(t) - \int_0^t & I(s)ds \text{ is a } \mathcal{F}_t^N\text{-Martingale.}
\end{aligned}
\tag{7}
$$

Note that first, with a stopping time τ_M such that $\sup_{0 \le t \le T} \sup_{k=1}^N n_k^N(t \wedge \tau_M) 1_{\{\tau_M > 0\}} < M$, we obtain that $n_k^N(t \wedge \tau_M) - \int_0^{t \wedge \tau_M} I(s)ds$ is a $\mathcal{F}_t^N$-Martingale

for all $M > 0$. Equation (7) will then follow from the proof of Lemma 3.3. Our approximating Markov jump process will be

$$X^N(t,r) := X^{N,l}(t,r) := \frac{n_k^N(t)}{l},\ r \in [\frac{k-1}{N}, \frac{k}{N}),\ \text{(and periodic extension)}. \tag{8}$$

Let H^N be the $L^2(0,1)$-subspace of step functions on $[0,1)$ which are constant on the intervals $[\frac{k-1}{N}, \frac{k}{N})$. Define the orthogonal projection $P_N : L^2(0,1) \to H^N$ by

$$P_N f(r) = N \int_{\frac{k-1}{N}}^{\frac{k}{N}} f(x)dx \text{ for } r \in [\frac{k-1}{N}, \frac{k}{N}), \tag{9}$$

and introduce the discrete derivatives

$$\begin{aligned} \nabla_N^{\pm} f(r) &= \pm N[P_N f(r \pm N^{-1}) - P_N f(r)], \\ \Delta_N f(r) &= \nabla_N^- \nabla_N^+ f(r) = \nabla_N^+ \nabla_N^- f(r) \\ &= N^2[P_N f(r + N^{-1}) - 2P_N f(r) + P_N f(r - N^{-1})]. \end{aligned} \tag{10}$$

From (8) and (7) follows that

$$X^N(t) = X^N(0) + \int_0^t \Delta_N X^N(s) + \nabla_N^+ F_N(X^N(s))ds + Z^N(t) \tag{11}$$

where

$$F_N : \left\{ \begin{array}{l} H^N \mapsto H^N \\ X \to \frac{1}{3}[(X(\cdot))^2 + X(\cdot)X(\cdot - N^{-1}) + (X(\cdot - N^{-1}))^2] \end{array} \right. \tag{12}$$

and $Z^N(t)$ is an H^N-valued martingale for $\mathcal{F}_t^N$. In mild form this becomes

$$X^N(t) = e^{\Delta_N t} X^N(0) + \int_0^t e^{\Delta_N (t-s)} \nabla_N^+ F_N(X^N(s))ds + Y^N(t) \tag{13}$$

where

$$Y^N(t) = \int_0^t e^{\Delta_N (t-s)} dZ^N(s) \tag{14}$$

(note that Z^N is of bounded variation P-a.s. because it is piecewise absolutely continuous). For technical reasons we assume that N is odd.

We obtain the following result (for the definition of the spaces see Definition 3.2). $\langle \cdot, \cdot \rangle$ denotes the dual pairing and simultaneously the $L^2(0,1)$-scalar product.

Theorem 2.1 *Let X^N be the process defined by (8) with deterministic initial condition $0 \le X^N(0) \in H^N$, such that for arbitrary $\alpha \in (0, \frac{1}{2})$*

$$\|X^N(0) - \psi_0\|_{H^\alpha(0,1)} \stackrel{N\to\infty}{\longrightarrow} 0,$$

where $0 \le \psi_0 \in H^\alpha(0,1)$ is the initial condition of (3). Moreover assume $l \ge qN$ for arbitrary $q > 0$.

Then there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, subsequences $(N_k)_{k\in\mathbb{N}}$ and $(l_k)_{k\in\mathbb{N}}$, and H^{N_k}-valued processes $\tilde{X}^{N_k}$, $\tilde{Y}^{N_k}$, and $\tilde{Z}^{N_k}$ on this probability space. The common distribution of $\tilde{X}^{N_k}$, $\tilde{Y}^{N_k}$, and $\tilde{Z}^{N_k}$ equals the common distribution of X^{N_k}, Y^{N_k}, and Z^{N_k}, for each $k \in \mathbb{N}$. There exist processes ψ in $C(0,T,L^2(0,1))$, $\tilde{Y}$ in $C(0,T,H^{\alpha_1}(0,1))$, and M in $C(0,T,H^{\alpha_2}(0,1))$ with $\alpha_1 < \frac{1}{2}$ and $\alpha_2 < -\frac{1}{2}$. M is a martingale w.r.t. $(\sigma(\psi(s), s \leq t))_t$. We obtain

$$(\tilde{X}^{N_k}, \tilde{Y}^{N_k}, \tilde{Z}^{N_k}) \stackrel{k\to\infty}{\longrightarrow} (\psi, \tilde{Y}, M)$$

$\tilde{P}$-almost sure in $L^2(0,T,L^2(0,1)) \times D(0,T,H^{\alpha_1}(0,1)) \times D(0,T,H^{\alpha_2}(0,1))$. The equation

$$\psi(t) = e^{t\Delta}\psi(0) + \int_0^t e^{(t-s)\Delta}\nabla(\psi(s))^2 ds + \tilde{Y}(t) \tag{15}$$

is fulfilled $\tilde{P}$-a.s. in $C(0,T,L^2(0,1))$ where $\tilde{Y}(t) = \int_0^t e^{(t-s)\Delta} dM(s)$ $\tilde{P}$-a.s. in $C(0,T,H^{\alpha_1}(0,1))$. Here $e^{t\Delta}$ denotes the semigroup defined by the Laplacian Δ with periodic boundary conditions. The equation

$$\langle \psi(t), \varphi \rangle = \langle \psi(0), \varphi \rangle + \int_0^t \langle \Delta\psi(s) + \nabla(\psi(s))^2, \varphi \rangle ds + \langle M(t), \varphi \rangle \tag{16}$$

holds $\tilde{P}$-a.s. in $C(0,T,\mathbb{R})$ where $\varphi \in C^{\alpha_3}_{per}(0,1)$ with $\alpha_3 > \frac{5}{2}$ and $\langle M(t), \varphi \rangle = \int_0^t \int_0^1 \sqrt{\gamma\psi(s,x)}\varphi(x) dW(s,x)$ where W is a certain space-time-white noise. In this sense, M can be represented as

$$M(t) = \int_0^t \sqrt{\gamma\psi(s)} dW(s).$$

Remark 2.2 *The proof of the theorem will be given in the next section in Lemmata 3.9, 3.10, 3.11, 3.12, and 3.13. We have not tried to prove uniqueness of a solution of Equation (3), see [11]. We can generalize Theorem 2.1 to random initial conditions. Note that the l-dependence of the quantities in the theorem is suppressed in the notation.*

3 Proofs

Definition 3.1 *(i) Eigenfunctions of Δ : Set $\varphi_0(r) := 1$ and*

$$\begin{aligned} \varphi_n(r) &:= \sqrt{2}\sin(2\pi n r) \text{ for } n \in \mathbb{N}, \\ \varphi_n(r) &:= \sqrt{2}\cos(2\pi n r) \text{ for } n \in \mathbb{Z} \setminus \mathbb{N}_0. \end{aligned} \tag{17}$$

The eigenfunctions of Δ with periodic boundary conditions on $(0,1)$ corresponding to the eigenvalues $\lambda_n = -4\pi^2 n^2$ are given by the complete orthonormal system $(\varphi_n)_{n\in\mathbb{Z}} \subset L^2(0,1)$.

(ii) Eigenfunctions of Δ_N : Let

$$\varphi_{n,N}(r) := \varphi_n(\frac{k-1}{N}) \text{ for } r \in [\frac{k-1}{N}, \frac{k}{N}), \tag{18}$$

where $k = 1, \dots, N, n = -\frac{N-1}{2}, \dots, \frac{N-1}{2}$ and N is assumed to be odd. According to [10], $(\varphi_{n,N})_n$ form a complete orthonormal system in the space $H^N \subset L^2(0,1)$ of piecewise constant functions (defined in Section 2). They are the eigenfunctions of Δ_N corresponding to the eigenvalues $\beta_{n,N} = -2N^2(1-\cos(\frac{2\pi n}{N}))$. There are constants $0 < c_1 < c_2$ with

$$c_1|\lambda_n| < |\beta_{n,N}| < c_2|\lambda_n| \tag{19}$$

for all $n = -\frac{N-1}{2}, \dots, \frac{N-1}{2}$.

(iii) Projection operators: P_N is the $L^2(0,1)$-orthogonal projection on H^N and P_n the $L^2(0,1)$-orthogonal projection on span$\{\varphi_k, k = -n, \dots, n\}$.

Definition 3.2 *We define the usual Sobolev spaces of order $\alpha \in \mathbb{R}$ with periodic boundary conditions by*

$$H^\alpha(0,1) := \{f = \sum_{n\in\mathbb{Z}} \alpha_n \varphi_n, (\alpha_n)_{n\in\mathbb{Z}} \subset \mathbb{R} \text{ with } \|f\|_{H^\alpha(0,1)} < \infty\}$$

where $\|f\|^2_{H^\alpha(0,1)} := \sum_{n\in\mathbb{Z}} \alpha_n^2 (1-\lambda_n)^\alpha$. Similarly we set

$$H^\alpha_N(0,1) := \{f \in H^N : \|f\|^2_{H^\alpha_N(0,1)} := \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} \langle f, \varphi_{n,N}\rangle^2 (1-\beta_{n,N})^\alpha < \infty\}.$$

Set $\delta X(t) = X(t) - X(t-) = X(t) - \lim_{s<t, s\to t} X(s)$. Then the following are $\mathcal{F}^N_t$-martingales.

$$\begin{aligned} Z^N_D(t) &:= \sum_{s\le t} \delta X^N_D(s) - \int_0^t \Delta_N X^N(s) + \nabla^+_N F_N(X^N(s)) ds, \\ Z^N_B(t) &:= \sum_{s\le t} \delta X^N_B(s), \end{aligned} \tag{20}$$

where $\delta X^N_D \in H^N$ is a jump caused by diffusion and δX^N_B is a jump by birth or death. The proof is similar to [5]. Moreover,

$$\begin{aligned} &\langle Z^N_D(t), f\rangle^2 - \frac{1}{Nl}\int_0^t \langle X^N(s), (\nabla^+_N f)^2\rangle + \langle X^N(s) + \frac{1}{N}F_N(X^N(s)), (\nabla^-_N f)^2\rangle ds, \\ &\langle Z^N_B(t), f\rangle^2 - \gamma \int_0^t \langle X^N(s), f^2\rangle ds \end{aligned} \tag{21}$$

are $\mathcal{F}^N_t$-martingales, $f \in H^N$.

Lemma 3.3 *Let the conditions of Theorem 2.1 be fulfilled. Then with $Y^N_B(t) = \int_0^t e^{\Delta_N(t-s)} dZ^N_B(s)$,*

$$\sup_N P(\|Y^N_B\|_{L^\infty(0,T,H^{\alpha_1}_N)} \ge \tilde{R}) \overset{\tilde{R}\to\infty}{\longrightarrow} 0,$$

for $\alpha_1 < \frac{1}{2}$.

Proof: The proof follows [10], Lemma 3.2. We therefore only give a brief sketch of the idea. Let

$$R(t) = \int_0^t e^{(t-s)\Delta_N} dZ_B^N(s \wedge \tau)$$

where $\tau = \tau_N := \inf\{t \in [0,T] : \langle X^N(t), 1\rangle \geq \rho\}$. Since $P(\|Y_B^N\|_{L^\infty(0,T,H_N^{\alpha_1})} \geq \tilde{R}) \leq P(\|R\|_{L^\infty(0,T,H_N^{\alpha_1})} \geq \tilde{R}) + P(\tau_N < T)$ we have to show $\sup_N P(\tau_N < T) \overset{\rho\to\infty}{\longrightarrow} 0$ and for fixed $\rho > 0$, $\sup_N P(\|R\|_{L^\infty(0,T,H_N^{\alpha_1})} \geq \tilde{R}) \overset{\tilde{R}\to\infty}{\longrightarrow} 0$. Let now $\rho > 0$ be fixed and define for $m \neq 0$ and $u \in [0,t]$

$$M(u) = |m| \int_0^u e^{\beta_{m,N}(t-s)} d\langle Z_B^N(s \wedge \tau), \varphi_{m,N}\rangle.$$

This is a mean-zero-martingale with $M(t) = |m|\langle R(t), \varphi_{m,N}\rangle$ and $|\delta M(u)| \leq 1$. The predictable quadratic variation process $<< M >>$ fulfills $<< M >> (u) \leq c\gamma\rho$, see (21). Lemma 4.4 of [7] yields $E[\exp(M(t))] \leq \exp(\frac{3}{2}c\gamma\rho)$ whence

$$P(m^{2\alpha_1}\langle R(t), \varphi_{m,N}\rangle^2 \geq m^{-2r}) \leq c(\gamma\rho)\exp(-|m|^{1-r-\alpha_1})$$

Because for $\alpha_1 < \frac{1}{2}$ there exists $r > \frac{1}{2}$ with $\alpha_1 + r < 1$ such that $\sum_{m\in\mathbb{Z}\setminus\{0\}} |m|^{-2r} < \infty$ and $\sum_{m\in\mathbb{Z}\setminus\{0\}} m^2 \exp(-|m|^{1-r-\alpha_1}) < \infty$, we obtain as in [10]

$$\sup_N P(\sup_{t\leq T} \|R(t)\|_{H_N^{\alpha_1}} \geq \tilde{R}) \leq c(\gamma,\rho,T,\alpha_1) \sum_{m\in\mathbb{Z}\setminus\{0\}} m^2 \exp(-c(T)|m|^{1-r-\alpha_1}\tilde{R})$$
$$+ \sup_N P(\sup_{t\leq T}\langle R(t), 1\rangle \geq \frac{\tilde{R}}{c}) \overset{\tilde{R}\to\infty}{\longrightarrow} 0$$

where the last term is treated similarly to the others. For the assertion of the Lemma it therefore remains to show $\sup_N P(\tau_N < T) \overset{\rho\to\infty}{\longrightarrow} 0$ which in turn follows from $E[\sup_{t\leq T}\langle X^N(t), 1\rangle] \leq c$ uniformly in N : From $\langle \Delta_N X^N(t) + \nabla_N^+ F_N(X^N(t)), 1\rangle = 0$ we conclude $\langle X^N(t), 1\rangle = \langle X^N(0), 1\rangle + \langle Z^N(t), 1\rangle$ and by the Jensen and maximal inequality and (20)

$$E[\sup_{t\leq T}\langle X^N(t\wedge\tau), 1\rangle] = E[\sup_{t\leq T}\langle Z^N(t\wedge\tau), 1\rangle] + \langle X^N(0), 1\rangle$$
$$\leq 2\sup_{t\leq T}\sqrt{E[\langle Z^N(t\wedge\tau), 1\rangle^2]} + \langle X^N(0), 1\rangle$$
$$\leq 4\sup_{t\leq T}\sqrt{E[\langle Z_D^N(t\wedge\tau), 1\rangle^2] + E[\langle Z_B^N(t\wedge\tau), 1\rangle^2]} + \langle X^N(0), 1\rangle \leq$$

since $\langle Z_D^N(t\wedge\tau), 1\rangle = 0$ a.s, we continue using Equation (21) and $E[\langle Z^N(t\wedge\tau), 1\rangle] = 0$

$$\leq 4\sup_{t\leq T}\sqrt{\gamma\int_0^{t\wedge\tau}\langle X^N(0), 1\rangle + E[\langle Z^N(s\wedge\tau), 1\rangle]ds} + \langle X^N(0), 1\rangle \leq c$$

uniformly in N. □

Lemma 3.4 *Let the conditions of Theorem 2.1 be fulfilled. Then with* $Y_D^N(t) = \int_0^t e^{\Delta_N(t-s)} dZ_D^N(s)$,

$$\sup_N P(\|Y_D^N\|_{L^\infty(0,T,H_N^{\alpha_1})} \geq \tilde{R}) \overset{\tilde{R}\to\infty}{\longrightarrow} 0,$$

for $\alpha_1 < \frac{1}{2}$.

Proof: We proceed as in the proof of Lemma 3.3. Due to (21) we obtain with the notation of this proof:

$$<< M >> (u) = \frac{m^2}{Nl} \int_0^{u\wedge\tau} \exp(2\beta_{m,N}(t-s))$$
$$\times \Big(\langle X^N(s), (\nabla_N^+ \varphi_{m,N})^2 \rangle + \langle X^N(s) + \frac{1}{N} F_N(X(s)), (\nabla_N^- \varphi_{m,N})^2 \rangle \Big) ds \leq$$

by $\|\nabla_N^+ \varphi_{m,N}\|_{L^\infty} \leq cm$

$$\leq c\frac{m^4}{Nl} \int_0^{u\wedge\tau} \exp(2\beta_{m,N}(t-s)) \Big(\|X^N(s)\|_{L^1} + \frac{1}{N} \|X^N(s)\|_{L^2}^2 \Big) ds \leq$$

by $\|X^N(s)\|_{L^2}^2 \leq N\|X^N(s)\|_{L^1}^2$

$$\leq c\frac{m^2}{Nl}(\rho + \rho^2) \leq c(\rho).$$

We can continue similarly to the proof of Lemma 3.3. □

Lemma 3.5 *Under the assumptions of Theorem 2.1, the family of the probability distributions of* Y_B^N *is tight on* $D(0,T,H^{\alpha_1}(0,1))$.

Proof: We again follow the proof of [10], Lemma 3.3 and first show that the distributions of Z_B^N are tight on $D(0,T,H^{\alpha_2}(0,1))$. We verify Condition (a) in Theorem 37.2 in [18] and (8.33) and (8.29) ibid. Let $\alpha_2 < \tilde{\alpha} < -\frac{1}{2}$ and $\Gamma_\eta := B_R(0) \subset H^{\tilde{\alpha}}(0,1) \subset\subset H^{\alpha_2}(0,1)$. Then by (21)

$$P(Z_B^N(t) \in \Gamma_\eta) \geq 1 - P(\|Z_B^N(t)\|_{H_N^{\tilde{\alpha}}}^2 > R^2)$$
$$\geq 1 - \frac{\gamma}{R^2} \sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} \int_0^t E[\langle X^N(s), \varphi_{m,N}^2 \rangle](1-\beta_{m,N})^{\tilde{\alpha}} ds \geq 1 - \eta$$

for sufficiently large R, see the proof of Lemma 3.3. Analogously, for $0 \leq t \leq T$ and $0 \leq u \leq \tilde{\delta}$ for some $\tilde{\delta} \in (0,1)$, $E[\|Z_B^N(t+u) - Z_B^N(t)\|_{H^{\tilde{\alpha}}(0,1)}^2 | \mathcal{F}_t^N] \leq c\tilde{\delta}$.

This shows the existence of $K_\eta \subset\subset D(0,T,H^{\alpha_2}(0,1))$ with $P(Z_B^N \in K_\eta) \geq 1-\eta$. By Lemma 4.6, $K_\eta \subset\subset L^p(0,T,H^{\alpha_2}(0,1))$ for all $p > 1$. According to Lemma 4.5,

$$\{\int_0^t e^{(t-s)\Delta_N} \Delta_N Z_B^N(s) ds \in C(0,T,H^{\alpha_2-\epsilon}(0,1)) | Z_B^N \in K_\eta\}$$
$$\subset\subset C(0,T,H^{\alpha_2-\epsilon}(0,1)) \subset D(0,T,H^{\alpha_2-\epsilon}(0,1))$$

for all $\epsilon > 0$. Because P_n is continuous from $D(0,T,H^{\alpha_2-\epsilon}(0,1))$ into $D(0,T,H^{\alpha_1}(0,1))$, and because of

$$Y_B^N(t) = Z_B^N(t) + \int_0^t e^{(t-s)\Delta_N}\Delta_N Z_B^N(s)ds, \tag{22}$$

see [10], the distributions of $P_n Y_B^N$ are tight on $D(0,T,H^{\alpha_1}(0,1))$ for fixed $n \in \mathbb{N}$. The assertion of the Lemma then follows from Problem 18, Chapter 3, in [18] and the fact that for all $\epsilon > 0$ there exist a $n \in \mathbb{N}$ such that $P(\|(I-P_n)Y_B^N\|_{L^\infty(0,T,H^{\alpha_1}(0,1))} \geq \epsilon) \leq \epsilon$ uniformly in N. This can easily be deduced from Lemma 3.3 and Lemma 4.2. □

Lemma 3.6 *Let the requirements of Theorem 2.1 be true. Then* $Y_D^N \to 0$ *for* $N \to \infty$ *in* $L^\infty(0,T,L^2(0,1))$ *in probability.*

Proof: According to Lemma 3.4 it suffices to show $P_n Y_D^N(\cdot \wedge \tau) \to 0$ for $N \to \infty$ in the $L^\infty(0,T,L^2(0,1))$-norm in probability for any fixed n, where τ is the stopping time from the proof of Lemma 3.3. Proving this can be done as in [6]. □

Let

$$u^N = X^N - Y^N. \tag{23}$$

By (13), $u^N : [0,T] \to H^N$ is continuous. Moreover, between two jumps, we obtain

$$\frac{\partial}{\partial t}u^N(t) = \Delta_N u^N(t) + \nabla_N^+ F_N(u^N + Y^N)(t). \tag{24}$$

Lemma 3.7 *With constants c independent of N, u^N, and Y^N we obtain the following a-priori-estimates:*

$$\begin{aligned}\|u^N(t)\|^2_{L^2(0,1)} \leq{}& \exp\Big(c\int_0^t (1 + \|Y^N(s)\|^{8/3}_{L^4(0,1)})ds\Big)\\ &\times\Big(\|X^N(0)\|^2_{L^2(0,1)} + c\int_0^t \|Y^N(s)\|^4_{L^4(0,1)}ds\Big) =: f(t)\end{aligned}$$

and

$$\begin{aligned}&\int_0^T \|\nabla_N^- u^N(t)\|^2_{L^2(0,1)}dt\\ &\leq \int_0^T \Big(cf(t)(1 + \|Y^N(t)\|^{8/3}_{L^4(0,1)}) + c\|Y^N(t)\|^4_{L^4(0,1)}\Big)dt + \|X^N(0)\|^2_{L^2(0,1)}.\end{aligned}$$

Proof: We apply a well-known procedure, see e.g. [14]. Both estimates follow from

$$\frac{\partial}{\partial t}\|u^N\|^2_{L^2} + \|\nabla_N^- u^N\|^2_{L^2} \leq c\|u^N\|^2_{L^2}\Big(1 + \|Y^N\|^{8/3}_{L^4}\Big) + c\|Y^N\|^4_{L^4}. \tag{25}$$

by an application of the Gronwall lemma. To obtain this, we multiply (24) with u^N and integrate over the spatial variable,

$$\frac{1}{2}\frac{\partial}{\partial t}\langle u^N, u^N\rangle - \langle \Delta_N u^N, u^N\rangle = -\langle \nabla_N^+ F_N(u^N + Y^N), u^N\rangle.$$

By partial integration $\langle \nabla_N^+ f, g\rangle = -\langle f, \nabla_N^- g\rangle$ for $f, g \in H^N$ we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\|u^N\|^2_{L^2(0,1)} + \|\nabla_N^- u^N\|^2_{L^2(0,1)} = S_1 + S_2 + S_3. \tag{26}$$

with

$$\begin{aligned}
S_1 &= -\frac{1}{3}\langle (u^N(\cdot))^2 + u^N(\cdot)u^N(\cdot - \frac{1}{N}) + (u^N(\cdot - \frac{1}{N}))^2, \nabla_N^- u^N\rangle,\\
S_2 &= -\frac{1}{3}\langle 2u^N(\cdot)Y^N(\cdot) + 2u^N(\cdot - \frac{1}{N})Y^N(\cdot - \frac{1}{N})\\
&\quad + u^N(\cdot)Y^N(\cdot - \frac{1}{N}) + u^N(\cdot - \frac{1}{N})Y^N(\cdot), \nabla_N^- u^N\rangle,\\
S_3 &= -\frac{1}{3}\langle (Y^N(\cdot))^2 + Y^N(\cdot)Y^N(\cdot - \frac{1}{N}) + (Y^N(\cdot - \frac{1}{N}))^2, \nabla_N^- u^N\rangle.
\end{aligned}$$

We now treat S_1, S_2, and S_3.

$S_1 = 0$, because

$$S_1 = -\frac{N}{3}\sum_{k=1}^{N}(u^N(\frac{k}{N}))^3 - (u^N(\frac{k-1}{N}))^3 = 0$$

due to periodic boundary conditions. This is the discrete equivalent to the standard trick $\int_0^1 u^2 \frac{\partial}{\partial x}u = \frac{1}{3}\int_0^1 \frac{\partial}{\partial x}u^3 = 0$ used when treating the Burgers equation with periodic boundary conditions. Note that for this result we impose the jump rate in (4). Otherwise we could have taken $N^2 n_k + \frac{N}{l}n_k^2$ as first rate in (4) which would have entailed $F_N(X) = X^2$ in (12) and therefore a "usual" deterministic Burgers equation in (13) and (24).

$S_2 + S_3$ can be bounded by $c\|u^N\|_{L^4}\|Y^N\|_{L^4}\|\nabla_N^- u^N\|_{L^2} + \|Y^N\|^2_{L^4}\|\nabla_N^- u^N\|_{L^2}$. Since $\|u^N\|_{L^4} \le c\|u^N\|_{H_N^{\frac{1}{4}}}$ due to a standard Sobolev imbedding ([1], Theorem 7.57) and Lemma 4.2, and due to Lemma 4.3,

$$\|u^N\|_{H_N^{\frac{1}{4}}} \le c\|u^N\|^{\frac{3}{4}}_{L^2}\|u^N\|^{\frac{1}{4}}_{H_N^1} \le c\|u^N\|_{L^2} + c\|u^N\|^{\frac{3}{4}}_{L^2}\|\nabla_N^- u^N\|^{\frac{1}{4}}_{L^2},$$

the crucial term in the bound of $S_2 + S_3$ is

$$c\|u^N\|^{\frac{3}{4}}_{L^2}\|\nabla_N^- u^N\|^{\frac{5}{4}}_{L^2}\|Y^N\|_{L^4} \le \frac{1}{6}\|\nabla_N^- u^N\|^2_{L^2} + c\|u^N\|^2_{L^2}\|Y^N\|^{\frac{8}{3}}_{L^4}.$$

This yields (25). □

Lemma 3.8 *Let the conditions of Theorem 2.1 be fulfilled. Then the family of the probability distributions of u^N is tight on $L^2(0,T,L^2(0,1))$.*

Proof: By the computation in the proof of Lemma 4.3 we have for all $\beta \in \mathbb{R}$ and $v^N \in H^N$, $\|\nabla_N^+ v^N\|^2_{H_N^\beta} \le c\|v^N\|^2_{H_N^{\beta+1}}$. A rough estimate now gives

$$\begin{aligned}
\|F_N(u^N + Y^N)\|^2_{H_N^{\beta+1}} &\le c\sum_{m=1}^{\frac{N-1}{2}}\langle F_N(u^N + Y^N), 1\rangle^2(1 - \beta_{m,N})^{1+\beta}\\
&\le c\|u^N + Y^N\|^4_{L^2}\sum_{m=1}^{\frac{N-1}{2}}(1 - \beta_{m,N})^{1+\beta} \le c\|u^N + Y^N\|^4_{L^2}
\end{aligned}$$

for $\beta < -\frac{3}{2}$. Therefore

$$\sup_N P\Big(\int_0^T \|\nabla_N^+ F_N(u^N + Y^N)\|^2_{H_N^\beta} dt \ge R \Big)$$
$$\le \sup_N P\Big(\int_0^T \|u^N + Y^N\|^4_{L^2} \ge \frac{R}{c} \Big) \overset{R\to\infty}{\longrightarrow} 0$$

according to Lemmata 3.7, 3.3 with Lemma 4.2, and Lemma 3.4. Let $(R_N h)(t) = \int_0^t e^{(t-s)\Delta_N} h(s) ds$ for $h \in H^N$ and

$$\begin{aligned}\Xi(R) \;=\; & \{u^N \in C(0,T,H^\beta) \cap L^2(0,T,H_N^1): \\ & \|\nabla_N^+ F_N(u^N + Y^N)\|_{L^2(0,T,H_N^\beta)} \le R, \|\nabla_N^+ u^N\|_{L^2(0,T,L^2)} \le R, \\ & u^N(t) = e^{t\Delta_N} X^N(0) + R_N(\nabla_N^+ F_N(u^N + Y^N))(t)\}.\end{aligned}$$

Then by the equivalence of the norms in H^β and H_N^β for $\beta \le 0$,

$$\begin{aligned}& P(\Xi(R)) \\ & \ge 1 - P(\|\nabla_N^+ F_N(u^N + Y^N)\|_{L^2(0,T,H^\beta)} \ge R) - P(\|\nabla_N^+ u^N\|_{L^2(0,T,L^2)} \ge R) \\ & \ge 1 - \epsilon\end{aligned}$$

for $R = R(\epsilon)$ according to Lemmata 3.7, 3.3 with Lemma 4.2, and Lemma 3.4. Moreover, $\Xi(R)$ is compact in $L^2(0,T,L^2)$ according to Lemmata 4.5 and 4.4. □

Lemma 3.9 *Let the conditions of Theorem 2.1 be fulfilled. Then there exist subsequences $(N_k)_{k\in\mathbb{N}}$ and $(l_k)_{k\in\mathbb{N}}$ and a probability measure μ such that in distribution on $L^2(0,T,L^2(0,1)) \times D(0,T,H^{\alpha_1}(0,1)) \times D(0,T,H^{\alpha_2}(0,1)) \times D(0,T,H^{\alpha_1}(0,1))$,*

$$(u^{N_k}, Y_B^{N_k}, Z_B^{N_k}, Y_D^{N_k}) \overset{k\to\infty}{\longrightarrow} \mu$$

for $\alpha_1 < \frac{1}{2}$ and $\alpha_2 < -\frac{1}{2}$.

Proof: See Lemma 3.6, Lemma 3.5 and its proof, and Lemma 3.8. The tightness of the family of the probability distributions of Y_D^N is shown as in Lemma 3.5 using an estimate $E[\|X^N(t)\|^2_{L^2(0,1)}] \le c_T N$ derived similarly to the proof of Lemma 3.3. The conclusion follows from the theorem of Prokhorov, e.g. [18], Chapter 3. □

Lemma 3.10 *Let the requirements of Theorem 2.1 be fulfilled. There exists a probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde P)$ and processes $\tilde u$ in $L^2(0,T,L^2(0,1))$, $\tilde Y_B$ in $D(0,T,H^{\alpha_1}(0,1))$, and M in $D(0,T,H^{\alpha_2}(0,1))$ with $\alpha_1 < \frac{1}{2}$ and $\alpha_2 < -\frac{1}{2}$. $(\tilde u, \tilde Y_B, M, 0)$ has the law defined in Lemma 3.9. There exist processes $\tilde u^{N_k}$, $\tilde Y_B^{N_k}$, $\tilde Y_D^{N_k}$, and $\tilde Z_B^{N_k}$ on this probability space such that the common distribution of $\tilde u^{N_k}$, $\tilde Y_B^{N_k}$, $\tilde Y_D^{N_k}$, and $\tilde Z_B^{N_k}$ equals the common distribution of u^{N_k}, $Y_B^{N_k}$, $Y_D^{N_k}$, and $Z_B^{N_k}$, for each $k \in \mathbb{N}$. Moreover, in $L^2(0,T,L^2(0,1)) \times D(0,T,H^{\alpha_1}(0,1)) \times D(0,T,H^{\alpha_2}(0,1)) \times D(0,T,H^{\alpha_1}(0,1))$,*

$$(\tilde u^{N_k}, \tilde Y_B^{N_k}, \tilde Z_B^{N_k}, \tilde Y_D^{N_k}) \overset{k\to\infty}{\longrightarrow} (\tilde u, \tilde Y_B, M, 0),$$

$\tilde P$-almost surely.

Proof: This lemma follows from Lemma 3.9, Lemma 3.6 and the theorem of Skorohod, e.g. [18], Chapter 3. □

Lemma 3.11 *Let the assumptions of Lemma 3.10 be fulfilled. Then*

$$\tilde{u}(t) = e^{t\Delta}\psi_0 + \int_0^t e^{(t-s)\Delta}\nabla(\tilde{u}+\tilde{Y}_B)^2(s)ds \tag{27}$$

holds in $L^2(0,T,L^2(0,1))$, $\tilde{P}$*-a.s.*

Proof: For simplicity we denote the subsequence $(N_k)_{k\in\mathbb{N}}$ in the Lemmata 3.9 and 3.10 by N, suppress the tilde, and replace $L^2(0,1)$ by L^2, e.g., in this proof. We give the proof in several steps.

(i) $\|F_N(u^N)\|_{L^1} \le \|u^N\|^2_{L^2}$, see (12).

(ii) $\|F_N(u^N) - u^2\|_{L^1} \le \|u^N - u\|_{L^2}(\|u^N\|_{L^2} + \|u\|_{L^2})$ since

$$\begin{aligned}
&\int_0^1 \frac{1}{3}\Big((u^N(x))^2 + u^N(x)u^N(x-N^{-1}) + (u^N(x-N^{-1}))^2\Big) - \Big(u(x)\Big)^2 dx \\
\le\ & \frac{1}{3}\Big(\|u^N+u\|_{L^2}\|u^N-u\|_{L^2} + \|u^N+u\|_{L^2}\|u^N-u\|_{L^2} \\
&+\|u^N\|_{L^2}\|u^N(\cdot)-u^N(\cdot-N^{-1})\|_{L^2} + \|u^N+u\|_{L^2}\|u^N-u\|_{L^2}\Big) \\
\le\ & \|u^N+u\|_{L^2}\|u^N-u\|_{L^2} + \frac{1}{N}\|u^N\|_{L^2}\|\nabla_N^- u^N\|_{L^2}.
\end{aligned}$$

(iii) $\|\nabla\varphi - \nabla_N^-\varphi\|_{L^\infty} \le \frac{1}{N}\|\varphi\|_{H^{a_1}}$ for $a_1 > \frac{5}{2}$ because:

$$\begin{aligned}
&\|\nabla\varphi - \nabla_N^-\varphi\|_{L^\infty} \\
&= \sup_{k=1,\dots,N}\ \sup_{x\in[\frac{k-1}{N},\frac{k}{N})} |\varphi'(x) - N\int_{\frac{k-1}{N}}^{\frac{k}{N}} \frac{\varphi(y)-\varphi(y-N^{-1})}{N^{-1}}dy| \\
&= \sup_{k=1,\dots,N}\ \sup_{x\in[\frac{k-1}{N},\frac{k}{N})} |\varphi'(x) - \varphi'(\xi_{k,N})| \le \frac{c}{N}\|\varphi''\|_{L^\infty}
\end{aligned}$$

with some $\xi_{k,N} \in [\frac{k-1}{N},\frac{k}{N}]$.

(iv) If $u^N \to u$ in $L^2(0,T,L^2)$ then $\nabla_N^+ F_N(u^N) \to \nabla u^2$ in $L^1(0,T,H^{-a_1})$ for $a_1 > \frac{5}{2}$. For

$$\begin{aligned}
&\|\nabla_N^+ F_N(u^N) - \nabla u^2\|_{H^{-a_1}} = c \sup_{\|\varphi\|_{H^{a_1}}=1} \langle \nabla_N^+ F_N(u^N) - \nabla u^2, \varphi\rangle \\
&\le c \sup_{\|\varphi\|_{H^{a_1}}=1} \langle F_N(u^N), \nabla\varphi - \nabla_N^-\varphi\rangle + c \sup_{\|\varphi\|_{H^{a_1}}=1} \langle u^2 - F_N(u^N), \nabla\varphi\rangle \\
&\le \frac{c}{N}\|u^N\|^2_{L^2} + c\|u^N-u\|_{L^2}\Big(\|u^N\|_{L^2} + \|u\|_{L^2}\Big) + \frac{c}{N}\|u^N\|_{L^2}\|\nabla_N^- u^N\|_{L^2}
\end{aligned}$$

according to steps (i), (ii), and (iii).

(v) For fixed $n \in N$, $e^{\beta_{n,N}t} - e^{\lambda_n t} \to 0$ for $N \to \infty$ uniformly in $t \leq T$, because $|e^{\beta_{n,N}t} - e^{\lambda_n t}| \leq t|\beta_{n,N} - \lambda_n|$ and $|\beta_{n,N} - \lambda_n| = 4\pi n^2 |2\frac{1-\cos x}{x^2} - 1| \to 0$ for $x \to 0$ where $x = \frac{2\pi n}{N}$.

(vi) If h^N is bounded in $L^1(0,T,H^{-a_1})$ then for $a_2 > a_1$,

$$\|e^{\Delta_N t}h^N - e^{\Delta t}h^N\|_{L^1(0,T,H^{-a_2})} \to 0 \text{ for } N \to \infty.$$

Applying Lemma 4.1 we obtain

$$\begin{aligned}
&\|e^{\Delta_N t}h^N - e^{\Delta t}h^N\|^2_{H^{-a_2}} \\
&= \|\sum_{k\in\mathbb{Z}} \Big(\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} \langle h^N, \varphi_{n,N}\rangle\langle\varphi_{n,N}, \varphi_k\rangle(e^{\beta_{n,N}t} - e^{\lambda_k t})\Big)\varphi_k\|^2_{H^{-a_2}} \\
&\leq \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} \sum_{l\in\mathbb{Z}} (1-\lambda_{n+lN})^{-a_2}\langle h^N, \varphi_{n,N}\rangle^2 (a^2_{n+lN} + b^2_{n+lN})(e^{\beta_{n,N}t} - e^{\lambda_{n+lN}t})^2
\end{aligned}$$

by a consideration similar to the proof of Lemma 4.2, distinguishing the cases $l = 0, 1, \geq 2$

$$\begin{aligned}
&\leq c\sum_{n=1}^{\frac{N-1}{2}} \Big(\langle h^N, \varphi_{n,N}\rangle^2 + \langle h^N, \varphi_{-n,N}\rangle^2\Big)\Big((1-\lambda_n)^{-a_2}|e^{\beta_{n,N}t} - e^{\lambda_n t}|^2 \\
&\qquad + N^{2(a_1-a_2)}(1-\lambda_n)^{-a_1} + n^{-2a_2}\frac{N}{n}(1-\cos(\frac{2\pi n}{N})\Big) \\
&\leq c\Big(\max_{|n|\leq k_0} |e^{\beta_{n,N}t} - e^{\lambda_n t}|^2 + (1-\lambda_{k_0})^{a_1-a_2} + N^{2(a_1-a_2)} \\
&\qquad + \max_{|n|\leq k_0} |\frac{N}{n}(1-\cos(\frac{2\pi n}{N}))| + k_0^{2(a_1-a_2)}\Big)\|h^N\|^2_{H^{-a_1}}.
\end{aligned}$$

The claim follows from the boundedness of h^N and (v).

(vii) For $X_0^N \overset{N\to\infty}{\longrightarrow} \psi_0$ in H^α, $\alpha > 0$, we obtain $e^{\Delta_N t}X_0^N \to e^{\Delta t}\psi_0$ for $N \to \infty$ in $L^\infty(0,T,L^2)$.

(viii) For $u^N \overset{N\to\infty}{\longrightarrow} u$, in $L^2(0,T,L^2)$,

$$\int_0^t e^{(t-s)\Delta_N}\nabla_N^+ F_N(u^N)(s)ds \overset{N\to\infty}{\longrightarrow} \int_0^t e^{(t-s)\Delta}\nabla u^2(s)ds$$

in $L^1(0,T,H^{-a_2})$ because

$$\begin{aligned}
&\int_0^T \|\int_0^t e^{(t-s)\Delta_N}\nabla_N^+ F_N(u^N)(s)ds - \int_0^t e^{(t-s)\Delta}\nabla u^2(s)ds\|_{H^{-a_2}}dt \\
&\leq \int_0^T\int_0^t \|\Big(e^{(t-s)\Delta_N} - e^{(t-s)\Delta}\Big)\nabla_N^+ F_N(u^N)(s)\|_{H^{-a_2}} \\
&\qquad + \|e^{(t-s)\Delta}\Big(\nabla_N^+ F_N(u^N)(s) - \nabla u^2(s)\Big)\|_{H^{-a_2}}dsdt
\end{aligned}$$

The first summand in the integral tends to 0 because of (vi) and (iv), the second due to (iv). See Lemma 3.7 and Lemma 3.10.

(ix) $u^N + Y_B^N + Y_D^N \overset{N\to\infty}{\longrightarrow} u + Y_B$, $\tilde{P}$-a.s. in $L^2(0,T,L^2)$ by Lemmata 3.10 and 4.6. □

Lemma 3.12 *Under the requirements of Theorem 2.1, M and $\langle M, f\rangle^2 - \gamma\int_0^t \langle\psi(s), f^2\rangle ds$ are $(\sigma(\psi(s), s\le t))_t$-martingales, for all $f\in L^\infty(0,1)$, where $\psi := \tilde{u} + \tilde{Y}_B$, see Lemma 3.10. Moreover, $M\in C(0,T,H^{\alpha_2}(0,1))$, $\tilde{P}$-a.s. The quadratic variation process of $\langle M(t), f\rangle$ is given by $\gamma\int_0^t\langle\psi(s), f^2\rangle ds$, for all $f\in L^\infty(0,1)$.*

Proof: We follow the proof of Lemma 3.6 of [10]. According to the proof of Lemma 3.3 we have, for $m\in\mathbb{Z}$, that

$$\tilde{E}[\langle\tilde{Z}_B^{N_k}(t), \varphi_m\rangle^2] \le 2\gamma\tilde{E}\int_0^t \langle\tilde{X}^{N_k}(s), 1\rangle ds \le c$$

uniformly in $k\in\mathbb{N}$, where $\tilde{X}^{N_k} := \tilde{u}^{N_k} + \tilde{Y}_B^{N_k}$. From [18], Chapter 7, Problem 7, we infer that $\langle M, \varphi_m\rangle$ and then M are martingales w.r.t. the above filtration.

By the Burkholder inequality and the proof of Lemma 3.3,

$$\tilde{E}[\sup_{t\le T}\langle\tilde{Z}_B^{N_k}, f\rangle^4] \le c(\gamma, f)\tilde{E}[(\int_0^T \langle\tilde{X}^{N_k}(s), 1\rangle ds)^2] + c(X^{N_k}(0), f) \le c$$

uniformly in $k\in\mathbb{N}$. [18], Chapter 7, Problem 7 yields the second claim.

Moreover, M is continuous because $\|\delta\tilde{Z}_B^{N_k}(t)\|_{H^{\alpha_2}(0,1)} \le \frac{c}{N_k l_k} \to 0$ for $k\to\infty$.

The representation of the quadratic variation process of $\langle M(t), f\rangle$ follows from (21). □

Lemma 3.13 *For the quantities M and $\tilde{Y}_B$ defined in Lemma 3.10, $\tilde{P}$-a.s.,*

$$\tilde{Y}_B(t) = \int_0^t e^{(t-s)\Delta} dM(s)$$

holds in $C(0,T,H^{\alpha_1}(0,1))$. M can be represented as $M(t) = \int_0^t \sqrt{\gamma\psi(s)}dW(s)$ in the sense that $\langle M(t), \varphi\rangle = \int_0^t\int_0^1 \sqrt{\gamma\psi(s,x)}\varphi(x)dW(s,x)$ for all $\varphi\in C_{per}^\infty(0,1)$ where W is a certain space-time-white noise on a possibly again extended probability space. See also Chapter 2 of [35] for an introduction to integration w.r.t. martingale measures.

Moreover, $\tilde{u}\in C(0,T,L^2(0,1))$ and (27) holds in this space.

Proof: From (22) we infer that

$$\tilde{Y}_B^{N_k}(t) = \tilde{Z}_B^{N_k}(t) + \int_0^t e^{(t-s)\Delta_{N_k}}\Delta_{N_k}\tilde{Z}_B^{N_k}(s)ds,$$

and similar to the proof of Lemma 3.11 the right hand side converges to $M(t) + \int_0^t e^{(t-s)\Delta}\Delta M(s)ds$ in $D(0,T,H^{\alpha_2}(0,1)) + L^1(0,T,H^{\alpha_2-2}(0,1))$. By Lemma 3.12,

the equality $\tilde{Y}_B(t) = M(t) + \int_0^t e^{(t-s)\Delta}\Delta M(s)ds$ holds in $D(0,T,H^{\alpha_2}(0,1))$. Note that the stochastic integral $\int_0^t e^{(t-s)\Delta}dM(s)$ is well defined and has a version in $C(0,T,H^{\alpha_2}(0,1))$ according to [26]. For all $m \in \mathbb{Z}$ we have

$$\begin{aligned}\langle \tilde{Y}_B(t), \varphi_m\rangle &= \langle M(t), \varphi_m\rangle + \int_0^t e^{(t-s)\lambda_m}\lambda_m \langle M(s), \varphi_m\rangle ds \\ &= \int_0^t e^{(t-s)\lambda_m} d\langle M(s), \varphi_m\rangle = \langle \int_0^t e^{(t-s)\Delta} dM(s), \varphi_m\rangle\end{aligned}$$

in $C(0,T,\mathbb{R})$ by a stochastic partial integration formula whence $\tilde{Y}_B(t) = \int_0^t e^{(t-s)\Delta}dM(s) \in C(0,T,H^{\alpha_2}(0,1))$. Similarly to the proof of Lemma 3.3 we have that $P_n\tilde{Y}_B$ tends to $\tilde{Y}_B$ in $D(0,T,H^{\alpha_1}(0,1))$ and therefore $\tilde{Y}_B \in C(0,T,H^{\alpha_1}(0,1))$, $\tilde{P}$-a.s. This yields the first part of the claim.

The representation of the martingale M by a stochastic integral follows by [24].

Since $\tilde{u}^{N_k}$ is bounded in $L^2(0,T,H^1_N)$, it is bounded by Lemma 4.2 in $L^2(0,T,H^\alpha)$ for all $\alpha < \frac{1}{2}$. For possibly a subsubsequence this yields $\tilde{u}^{N_k} \to \tilde{u}$ in $L^2(0,T,H^\alpha)$ and a.s. in $[0,T]$ in H^α for all $\alpha < \frac{1}{2}$. From Lemma 3.7 we infer $\tilde{u} \in L^\infty(0,T,L^2)$ and the claim follows by the method of [14]. □

4 Auxiliary Results

Lemma 4.1 *Let $a_{0,N} = 1$ and $b_{0,N} = 0$ and*

$$a_{n,N} = \frac{N}{2\pi n}\sin(\frac{2\pi n}{N}),\ b_{n,N} = \frac{N}{2\pi n}(\cos(\frac{2\pi n}{N}) - 1), \tag{28}$$

for all $n \in \mathbb{Z}$. Then we obtain for $n = -\frac{N-1}{2}, \dots, \frac{N-1}{2}$ and $m \in \mathbb{Z}$

$$\begin{aligned}&\langle \varphi_{n,N}, \varphi_m\rangle = a_{m,N} \text{ for } m = \pm n + zN, z \in \mathbb{Z}, m \le 0, n \le 0,\\ &\langle \varphi_{n,N}, \varphi_m\rangle = \pm a_{m,N} \text{ for } m = \pm n + zN, z \in \mathbb{Z}, m > 0, n > 0,\\ &\langle \varphi_{n,N}, \varphi_m\rangle = -b_{m,N} \text{ for } m = \pm n + zN, z \in \mathbb{Z}, m > 0, n \le 0,\\ &\langle \varphi_{n,N}, \varphi_m\rangle = \pm b_{m,N} \text{ for } m = \pm n + zN, z \in \mathbb{Z}, m < 0, n > 0\end{aligned}$$

Otherwise, this scalar product is zero.

Proof: Elementary calculations. □

Lemma 4.2 *For $\alpha < \frac{1}{2}$ and $f \in H^\alpha_N(0,1)$, $\|f\|^2_{H^\alpha(0,1)} \le c\|f\|^2_{H^\alpha_N(0,1)}$ holds with a constant c independent of N and f.*

Proof: Since this is not proved in the references we know of, we sketch the proof. By Definition 3.2 and Lemma 4.1

$$\|f\|^2_{H^\alpha} \le \langle f, 1\rangle^2 + c\sum_{n=1}^{\frac{N-1}{2}} \langle f, \varphi_{n,N}\rangle^2 \Big(\sum_{l\in\mathbb{Z}} (a^2_{\pm n+lN,N} + b^2_{\pm n+lN,N})(1 - \lambda_{\pm n+lN})^\alpha\Big)$$

where

$$\sum_{l\in\mathbb{Z}}(a^2_{\pm n+lN,N}+b^2_{\pm n+lN,N})(1-\lambda_{\pm n+lN})^\alpha \le cN^2(1-\cos(\frac{2\pi n}{N}))\sum_{l\in\mathbb{Z}}(\pm n+lN)^{2(\alpha-1)}.$$

This can easily be estimated by $\frac{cN^2}{(1-2\alpha)n^2}(1-\cos(\frac{2\pi n}{N}))n^{2\alpha} \le c\frac{n^{2\alpha}}{1-2\alpha}$ which implies the claim. □

Lemma 4.3 *For $f \in H^1_N(0,1)$ we obtain with constants not depending on N and f*

$$\|f\|_{H_N^{\frac{1}{4}}(0,1)} \le c\|f\|^{\frac{3}{4}}_{L^2(0,1)}\|f\|^{\frac{1}{4}}_{H^1_N(0,1)}$$

and

$$\|f\|_{H^1_N(0,1)} \le c\|f\|_{L^2(0,1)} + c\|\nabla^-_N f\|_{L^2(0,1)}$$

Proof: The first inequality is an application of Hölder's inequality. For the second, we compute for $k=1,\dots,N$ and $m\neq 0$

$$\nabla^+_N\varphi_{m,N}(\frac{k}{N}) = 2\pi m(a_{m,N}\varphi_{-m,N}+b_{m,N}\varphi_{m,N})(\frac{k}{N})$$

This yields for $m>0$

$$\begin{aligned}&\langle f,\nabla^+_N\varphi_{m,N}\rangle^2+\langle f,\nabla^+_N\varphi_{-m,N}\rangle^2\\ &=(2\pi m)^2\Big(\langle f,\varphi_{m,N}\rangle^2+\langle f,\varphi_{-m,N}\rangle^2\Big)\Big(a^2_{m,N}+b^2_{m,N}\Big).\end{aligned}$$

By $a^2_{m,N}+b^2_{m,N}=(-\beta_{m,N})/(2\pi m)^2$ we easily deduce the assertion from Definition 3.2. □

We now show a compactness criterion which is in some sense the discrete equivalent to Theorem IV.4.1 in [34].

Lemma 4.4 *Let $(u^N)_{N\in\mathbb{N}}$ be relatively compact in $C(0,T,H^\beta(0,1))$ for some $\beta\le 0$ and $(\nabla^+_N u^N)_{N\in\mathbb{N}}$ be bounded in $L^2(0,T,L^2(0,1))$. Then $(u^N)_{N\in\mathbb{N}}$ is relatively compact in $L^2(0,T,L^2(0,1))$.*

Proof: Let $\hat{u}^N$ be the piecewise linear function that coincides with u^N at the points k/N, $k=0,\dots,N-1$. Then $\|\hat{u}^N-u^N\|^2_{L^2}=\frac{1}{3N^2}\|\nabla^+_N u^N\|^2_{L^2}$ and therefore $\|\hat{u}^N\|^2_{H^1}\le c\|u^N\|^2_{L^2}+c(1+\frac{1}{N^2})\|\nabla^+_N u^N\|^2_{L^2}$ and $\|\hat{u}^{N+M}-\hat{u}^N\|^2_{H^\beta}\le c\|u^{N+M}-u^N\|^2_{H^\beta}+\frac{c}{N^2}\|\nabla^+_{N+M}u^{N+M}\|^2_{L^2}+\frac{c}{N^2}\|\nabla^+_N u^N\|^2_{L^2}$. Hence we deduce from a classical interpolation inequality that $\forall\epsilon>0:\exists C_\epsilon>0:\exists N_0\in\mathbb{N}:\forall N\in\mathbb{N}:N>N_0:\forall M\in\mathbb{N}:M>0:$

$$\begin{aligned}\|u^{N+M}-u^N\|^2_{L^2(0,T,L^2)} \le \epsilon\Big(&\|\nabla^+_{N+M}u^{N+M}\|^2_{L^2(0,T,L^2)}+\|\nabla^+_N u^N\|^2_{L^2(0,T,L^2)}\Big)\\ &+C_\epsilon\|u^{N+M}-u^N\|^2_{L^2(0,T,H^\beta)}.\end{aligned}$$

□

Lemma 4.5 *Let $(R_Nh^N)(t)=\int_0^t e^{(t-s)\Delta_N}h^N(s)ds$ for $h^N\in H^N$. Then for $p>1,\gamma>0$ such that $1>\frac{1}{p}+\gamma$ and for $\beta\in\mathbb{R}$ such that $\beta+2\gamma<0$, $(R_Nh^N)_{N\in\mathbb{N}}$ is relatively compact in $C(0,T,H^{\beta+2\gamma}(0,1))$ if $(h^N)_{N\in\mathbb{N}}$ is bounded in $L^p(0,T,H^\beta(0,1))$.*

Proof: Here we adapt the method of [19]. We obtain by a standard argument that $\|e^{t\Delta_N} f\|_{H^{2\gamma+\beta}} \leq (1+\frac{c}{t^\gamma})\|f\|_{H^\beta}$ for $\beta \in \mathbb{R}$ and $\gamma > 0$ with a constant c not dependent on N and f. An application of Hölders inequality then yields for sufficiently small $\tilde{\epsilon} > 0$ that for all $t \leq T$

$$\|R_N h^N(t)\|_{H^{2\gamma+\beta+\tilde{\epsilon}}} \leq c(T,\tilde{\epsilon},\gamma,p)\|h^N\|_{L^p(0,T,H^\beta)}$$

whence $(R_N h^N(t))_{N\in\mathbb{N}}$ is relatively compact in $H^{2\gamma+\beta}(0,1)$ for all $t \leq T$.

Similarly we estimate for $s < t$

$$\|\int_s^t e^{(t-\tau)\Delta_N} h^N(\tau)d\tau\|_{H^{2\gamma+\beta}} \leq c(T,\tilde{\epsilon},\gamma,p)\|h^N\|_{L^p(0,T,H^\beta)}|t-s|^{1-\frac{1}{p}-\gamma}.$$

Because due to $\beta+2\gamma < 0$, the norms of $H^{\beta+2\gamma}(0,1)$ and $H_N^{\beta+2\gamma}(0,1)$ are equivalent, see [10], we obtain $\|\Delta_N e^{t\Delta_N} f\|_{H^{2\gamma+\beta}} \leq (1+\frac{c}{t^{\gamma+1}})\|f\|_{H^\beta}$ and

$$\begin{aligned}
&\|\int_0^s \left(e^{(t-\tau)\Delta_N} - e^{(s-\tau)\Delta_N}\right)h^N(\tau)d\tau\|_{H^{2\gamma+\beta}} \\
&= \|\int_0^s \int_{s-\tau}^{t-\tau} \Delta_N e^{\Delta_N \rho} d\rho h^N(\tau)d\tau\|_{H^{2\gamma+\beta}} \\
&\leq c\int_0^s \|h^N(\tau)\|_{H^\beta}\Big((t-s) + (s-\tau)^{-\gamma} - (t-\tau)^{-\gamma}\Big)d\tau \leq
\end{aligned}$$

by a technique used in [14], Appendix A,

$$\leq c(T,\tilde{\epsilon},\gamma,p)\|h^N\|_{L^p(0,T,H^\beta)}|t-s|^{1-\frac{1}{p}-\gamma}$$

for $|t-s| \leq 1$. This shows that $(R_N h^N)_{N\in\mathbb{N}}$ is equicontinuous in $H^{2\gamma+\beta}(0,1)$ on $[0,T]$. $\square$

Lemma 4.6 *Convergence in $D(0,T,H^\alpha(0,1))$ implies convergence in $L^p(0,T,H^\alpha(0,1))$, $p \geq 1$.*

Proof: First, $D(0,T,H^\alpha) \subset L^\infty(0,T,H^\alpha)$ algebraically due to the existence of left limits and right continuity in $D(0,T,H^\alpha)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $D(0,T,H^\alpha)$ which converges to f in that space. Then according to [18] there exists a sequence of strictly increasing Lipschitz continuous functions $(\rho_n)_{n\in\mathbb{N}}$ with $\rho_n(0) = 0$ and $\rho_n(T) = T$ such that $\lim_{n\to\infty}\sup_{t\leq T}|\rho_n(t) - t| = 0$ and $\lim_{n\to\infty}\sup_{t\leq T}\|f_n(t) - f(\rho_n(t))\|_{H^\alpha} = 0$. Because of 5.5.1 in [18], $f(\rho_n(t)) \to f(t)$ a.e. for $n \to \infty$ and therefore $\int_0^T \|f(t) - f(\rho_n(t))\|_{H^\alpha}^p dt$ due to the integrable bound $c\|f\|^p_{L^\infty(0,T,H^\alpha)}$. $\square$

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